CGWAS 2015
Tuesday Exercise: Binary Source Modelling

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1 Introduction

In the lecture we have seen how we can find approximate solutions to the relativistic two-body problem. Calculations in post-Newtonian theory are very complex and difficult to perform analytically. However, in the 0th-order approximation, we will recover the solution obtained from linearised gravity. The following exercises aim to improve your understanding of the approximation of the relativistic two-body problem and to learn how to compute the gravitational wave polarisation at 0th-order. We will also perform a small PN computation and will determine the duration as a function of the invariant velocity v.

2 Exercises

Attempt to solve as many exercises as possible. If you have problems, please don't hesitate and ask either a lecturer or your teaching assistant of the day for help.

1. Consider a circular binary system of total mass $m = m_1 + m_2$ in the (xy)-plane of a Cartesian coordinate system, centred around the origin. Each mass is located at a position $\vec{r}_A$ for $A = 1,2$; $\mu = m_1 m_2/m$ denotes the reduced mass. The orbital separation is given by $l = r_1 + r_2$. The T_{00}-component of the stress-energy-momentum tensor of the binary is given by

$$T_{00} = \sum_{A=1}^{2} m_A c^2 \delta(x^1 - x^1_A)\delta(x^2 - x^2_A)\delta(x^3), \tag{1}$$

   and the moment of interia is defined as

$$M_{ij} := \frac{1}{c^2}\int x^i x^j T^{00} d^3x \tag{2}$$

   Compute the leading-order gravitational waveform (plus and cross polarisations) in the TT-gauge for an observer located on the z-axis at a distance r from the binary.
   Hint: use the property of the Kronecker delta $p\delta(p) = 0$. In the TT-gauge the quadrupole moment $Q^{\mathrm{TT}}_{ij} = M^{\mathrm{TT}}_{ij}$. Since we are considering a circular binary, use polar coordinates to parameterise the motion, where the orbital phase is given by $\phi = \omega t$.
   What do you note about the gravitational frequency when you compare it to the orbital frequency?

2. For the same binary system as in 1., re-express the previously calculated gravitational wave polarisations in terms of the small expansion parameter (v/c). Find the leading-order expression for the orbital phase $\phi(t)$.
   Hint: Use the condition for a circular orbit $v = r\omega$ and Kepler's third law, $Gm = l^3\omega^2$.

3. In the Newtonian case, the orbital energy and the gravitational flux are

$$E = -\frac{1}{2}\nu\left(\frac{v}{c}\right)^2, \quad \mathscr{F} = \frac{32}{5}\nu^2\left(\frac{v}{c}\right)^{10}. \tag{3}$$

   Use the energy balance equation to determine the duration $t(v)$ of the inspiral in the limit $v_c \to \infty$, where v_c denotes the coalescence velocity.
   Hint:

$$\frac{dt}{dv} = -\frac{Gm}{c^3}\frac{1}{\mathscr{F}}\frac{dE(v)}{dv}. \tag{4}$$

   Further, determine the orbital phase. Hint: start with $\phi(t) = \omega t$.
   Compute the evolution df_{GW}/dt of the gravitational frequency $f_{\mathrm{GW}} = v^3/\pi Gm$.
   This will give you the *Newtonian chirp*: you will see that the frequency of the gravitational wave monotonically increases with time.

4. We now perform the same computation at 1.5PN order. To do so, we need the energy and the flux given up to 1.5.PN order. Using the energy balance equation (see Eq.(4)), determine the duration $t(v)$ as a post-Newtonian expansion in terms of power (v/c). The 1.5PN expressions for the flux is given by

$$\mathscr{F} = \frac{32}{5}\left(\frac{\mu}{m}\right)^2\left(\frac{v}{c}\right)^{10}\left[1 + \left(-\frac{1247}{336} - \frac{5}{4}\mu\right)\left(\frac{v}{c}\right)^2 + 4\pi\left(\frac{v}{c}\right)^3 + \mathscr{O}(v^4)\right], \tag{5}$$

   where $v \equiv (\pi M f)^{1/3}$ is the *invariant velocity*, $\nu = \mu/m$ is the symmetric mass ratio. The energy at 1.5PN is

$$E = \mu - \frac{1}{2}\mu\left(\frac{v}{c}\right)^2\left[1 + \left(-\frac{3}{4} - \frac{\nu}{12}\right)\left(\frac{v}{c}\right)^2 + \mathscr{O}(v^4)\right]. \tag{6}$$

   Hint: In the integral, re-expand the denominator before you integrate and use the relation $(1+\varepsilon)^{-1} \simeq 1+\varepsilon$. This particular integration procedure will yield the PN approximant *TaylorT2*. We will see in the exercises Wednesday how the results can then be used to determine the gravitational wave phase. Now that you have obtained the Newtonian chirp time as well as the 1.5PN chirp time, compute the duration difference for a canonical binary system of your choice to get a better understanding for the effect of including higher order PN corrections.