CGWAS 2015
Tuesday Exercise: Binary Source Modelling – Solutions

Patricia Schmidt

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1 Solutions

1. First, we write the positions of the masses in polar coordinates. Since the binary is located in the (xy)-plane, we find
$$\vec{r}_1 = r_1(\cos\phi, \sin\phi, 0), \quad \vec{r}_2 = r_2(\cos\phi, \sin\phi, 0) \tag{1}$$
Next, we express the distance from the origin of each mass in terms of the orbital separation l:
$$r_1 = \frac{lm_2}{m}, \quad r_2 = \frac{lm_1}{m}. \tag{2}$$
We now compute the non-vanishing components of M_{ij}:
$$M_{xx} = m_1 x_1^2 + m_2 x_2^2 = \mu l^2 \cos^2\phi = \frac{\mu l^2}{2}(1+\cos 2\phi), \tag{3}$$
$$M_{yy} = m_1 y_1^2 + m_2 y_2^2 = \mu l^2 \sin^2\phi = \frac{\mu l^2}{2}(1-\cos 2\phi), \tag{4}$$
$$M_{xy} \equiv M_{yx} = m_1 x_1 y_1 + m_2 x_2 y_2 = \mu l^2 \cos\phi \sin\phi = \frac{\mu l^2}{2}\sin 2\phi \tag{5}$$
We now compute their first and second derivatives:
$$\ddot{M}_{xx} = -2\mu l^2 \omega^2 \cos 2\phi, \tag{6}$$
$$\ddot{M}_{yy} = 2\mu l^2 \omega^2 \cos 2\phi, \tag{7}$$
$$\ddot{M}_{xy} = \ddot{M}_{yx} = -2\mu l^2 \omega^2 \sin 2\phi, \tag{8}$$
where we have used the standard relation between the orbital phase and frequency, $\omega = d\phi(t)/dt$. For an observer located on the z-axis at a distance r from the origin, we compute the asymptotic quadrupole waveform from the formula given in the lecture
$$h_{ij}^{\mathrm{TT}} = \frac{2G}{rc^4}\ddot{M}_{ij}. \tag{9}$$
From the above it follows that
$$h_{ij}^{TT} = -\frac{4G\mu l^2\omega^2}{c^4 r}\begin{pmatrix} \cos 2\phi & \sin 2\phi & 0 \\ \sin 2\phi & -\cos 2\phi & 0 \\ 0 & 0 & 0 \end{pmatrix}^{\mathrm{TT}} \tag{10}$$

We note that the expression above is already in the TT-gauge and therefore can directly read off the gravitational wave polarisations:

$$h_+ = -\frac{4G\mu l^2\omega^2}{c^4 r}\cos 2\phi, \tag{11}$$

$$h_\times = -\frac{4G\mu l^2\omega^2}{c^4 r}\sin 2\phi. \tag{12}$$

We note that, in the quadrupole approximation, the frequency of the gravitational wave is *twice* the orbital frequency.

2. Combining the condition of a circular orbit with Kepler's third law yields the velocity in terms of the orbital frequency

$$v = (Gm\omega)^{1/3}, \tag{13}$$

and in term of the orbital separation

$$v = \sqrt{\frac{Gm}{l}}. \tag{14}$$

Rewriting the polarisation from Exercise 1 in terms of the above expressions, we find:

$$h_+ = -\frac{4G\mu}{c^2 r}\left(\frac{v}{c}\right)^2\cos 2\phi, \tag{15}$$

$$h_\times = -\frac{4G\mu}{c^2 r}\left(\frac{v}{c}\right)^2\sin 2\phi \tag{16}$$

We find the orbital phase in terms of v by substituting ω, yielding:

$$\phi = \omega t = \left(\frac{v}{c}\right)^3\frac{c^3}{Gm}t \tag{17}$$

3. From the ODE given, we obtain t(v) by means of integration and inserting the flux and derivative of the energy:

$$t(v) = t_c + \frac{Gm}{c^3}\int_v^{v_c}\frac{5}{32\nu^2}\left(\frac{v'}{c}\right)^{-10}\left(-\frac{1}{2}\right)\nu^2\left(-\frac{v'}{c}\right)dv' \tag{18}$$

$$\Rightarrow t(v) = t_c - \frac{Gm}{c^3}\frac{5}{256\nu}\left(\frac{v}{c}\right)^{-8}, \tag{19}$$

where t_c denotes the *time of coalescence*.

The frequency evolution is then obtained via the following equation:

$$\frac{df_{\mathrm{GW}}}{dt} = \frac{df_{\mathrm{GW}}}{dv}\frac{dv}{dt} = \frac{3v^2}{Gm\pi}\frac{c^4}{Gm}\left(\frac{v}{c}\right)^9 \tag{20}$$

$$= \frac{96}{5}\pi^{8/3}\nu\left(\frac{Gm}{c^3}\right)^{5/3}f_{\mathrm{GW}}^{11/3}, \tag{21}$$

where we have used the definition of the invariant gravitational wave frequency f_{GW}.

4. See the hand-written notes for the solution of the 1.5PN chirp time.

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$$\epsilon = \mu - \frac{1}{2}\mu v^2 \left[1 + \left(-\frac{3}{4} - \frac{\nu}{12} \right) v^2 \right]$$

$$\frac{d\epsilon}{dv} = -\mu v - \frac{1}{\cancel{2}}\mu \underbrace{\left(-\frac{3}{4} - \frac{\nu}{12} \right)}_{=:A} v^3 \cdot \cancel{4}^{2} = -\mu v - 2\mu A v^3 =: \epsilon'$$

$$\mathcal{F} = \frac{32}{5}\left(\frac{\mu}{m}\right)^2 v^{10} \left[1 + \underbrace{\left(-\frac{1247}{336} - \frac{5}{4}\mu \right)}_{=:B} v^2 + 4\pi v^3 \right]$$

$$\Rightarrow \frac{\epsilon'}{\mathcal{F}} = \frac{\frac{5}{32}\left(\frac{\mu}{m}\right)^{-2} v^{-10} \left(-\mu v - 2\mu A v^3 \right)}{\left(1 + B v^2 + 4\pi v^3 \right)}$$

Can write: $(1 + Bv^2 + 4\pi v^3)^{-1} \simeq 1 - Bv^2 - 4\pi v^3$
$(v \ll 1)$

$$\Rightarrow \frac{\epsilon'}{\mathcal{F}} = \frac{5}{32}\left(\frac{\mu}{m}\right)^{-2} v^{-10} \left(-\mu v - 2\mu A v^3 \right)\left(1 - Bv^2 - 4\pi v^3 \right)$$

$$= \frac{5}{32}\left(\frac{\mu}{m}\right)^{-2} v^{-10} \Big[-\mu v + \underline{\mu B v^3} + 4\pi\mu v^4 - \underline{2\mu A v^3} + 2\mu A B v^5 + 8\mu A \pi v^6 \Big]$$

Energy balance equation:

$$\int_t^{t_c} dt' = -\frac{G}{c^3} \underbrace{\int_v^{v_c \to \infty} \frac{\epsilon'}{\mathcal{F}} \, dv'}_{\text{solve the integral}}$$

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$$\Rightarrow \int \frac{\epsilon'}{F}\,dv = \frac{5}{32}\left(\frac{\mu}{m}\right)^{-2} \int \Big[-\mu v^{-9} + (\mu B - 2\mu A)\, v^{-7} + 4\pi\mu v^{-6} + 2\mu A B\, v^{-5} + 8\mu A \pi\, v^{-4} \Big]$$

$$= \frac{5}{32}\left(\frac{\mu}{m}\right)^{-2} \left[\frac{\mu v^{-8}}{8} - \frac{(\mu B - 2\mu A)\, v^{-6}}{6} - \frac{4\pi\mu v^{-5}}{5} + \dots \right]$$

Relative to the Newtonian piece we find at 1.5PN:

$$\int \frac{\epsilon'}{F}\,dv = \frac{5}{32}\left(\frac{\mu}{m}\right)^{-2} v^{-8} \left[\frac{\mu}{8} - \frac{(\mu B - 2\mu A)}{6} v^{2} - \frac{4\pi\mu}{5} v^{3} \right]$$

$$\Rightarrow \int_{t}^{t_c} dt' = -\frac{G}{c^3} \int_{v}^{v_c \to \infty} \frac{\epsilon'}{F}\,dv'$$

$$t_c - t = +\frac{G}{c^3}\left(\frac{5}{32}\right)\left(\frac{\mu}{m}\right)^{-2} v^{-8} \left[\frac{\mu}{8} + \frac{(\mu B - 2\mu A)}{6} v^{2} - \frac{4\pi\mu}{5} v^{3} \right]$$

$$\Rightarrow t = t_c - \frac{G}{c^3}\left(\frac{5}{32}\right)\underbrace{\left(\frac{\mu}{m}\right)^{-2}}\left(\frac{v}{c}\right)^{-8} \left[\frac{\mu}{8} + \frac{(\mu B - 2\mu A)}{6}\left(\frac{v}{c}\right)^{2} - \frac{4\pi\mu}{5}\left(\frac{v}{c}\right)^{3} \right]$$

After some algebra it follows:

$$\boxed{t(v) = t_c - \frac{Gm}{c^3 \nu}\left(\frac{5}{256}\right)\left(\frac{v}{c}\right)^{-8} \left[1 + \left(\frac{743}{252} + \frac{11}{3}\nu\right)\left(\frac{v}{c}\right)^{2} - \frac{32}{5}\pi\left(\frac{v}{c}\right)^{3} \right]}$$