Friday practicum: stochastic gravitational wave backgrounds
PTA solutions

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I. SOLUTIONS

1. For a pair of pulsars a and b, define a reference frame. Specifically, write down the unit vectors which point to the pulsars, $\hat{p}_a$, $\hat{p}_b$, the direction of GW propagation $\hat{\Omega}$ and the GW principal axes $\hat{m}$, $\hat{n}$, such that $\hat{m} \times \hat{n} = \hat{\Omega}$. *Hint, place pulsar a on the z-axis and b in the x-z plane*

   [Solution] For these investigations we use a particular reference frame, called the "computational frame", where one pulsar is placed along the z-axis and the second in the x-z plane, and the angle between the pulsars is ζ, as seen in Fig 1. Specifically, we write

$$\hat{p}_a = (0, 0, 1), \tag{1a}$$
$$\hat{p}_b = (\sin\zeta, 0, \cos\zeta), \tag{1b}$$
$$\hat{\Omega} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \tag{1c}$$
$$\hat{m} = (\sin\phi, -\cos\phi, 0), \tag{1d}$$
$$\hat{n} = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta), \tag{1e}$$

   This is indeed a convenient choice of geometry, as in this reference frame $F_a^\times = 0$, see below.

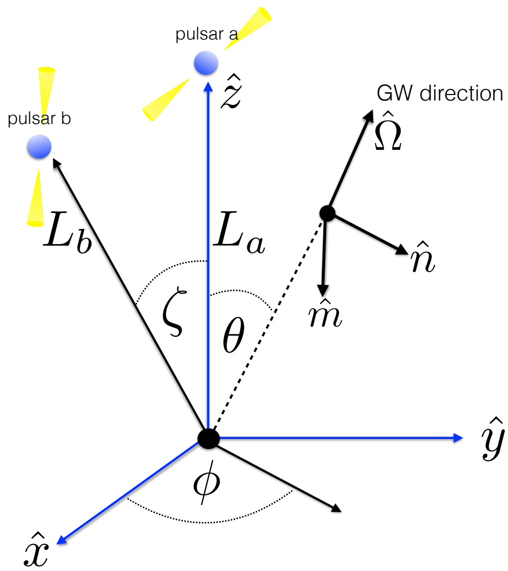

FIG. 1: Suggested geometry: pulsar a is on the z-axis at a distance L_a from the origin (solar system barycentre), pulsar b is in the x-z plane at a distance L_b from the origin making an angle ζ with pulsar a. $\hat{\Omega}$ is the direction of GW propagation with principal axes $\hat{m}$ and $\hat{n}$ such that $\hat{m} \times \hat{n} = \hat{\Omega}$. The polar and azimuthal angles are given by θ and ϕ, respectively.

2. Write down the general expression for the antenna beam pattern,

$$F^A(\hat{\Omega}) = \frac{1}{2}\frac{\hat{p}^i\hat{p}^j}{1+\hat{\Omega}\cdot\hat{p}} e^A_{ij}(\hat{\Omega}), \tag{2}$$

   where $A = +, \times$ is the GW polarization. The polarization tensors $e^+_{ij}(\hat{\Omega}) = \hat{m}_i\hat{m}_j - \hat{n}_i\hat{n}_j$, $e^\times_{ij}(\hat{\Omega}) = \hat{m}_i\hat{n}_j + \hat{n}_i\hat{m}_j$. For further reading, see [1, 2].

[Solution]
Using the definition of the polarization tensors above, we write

$$F^{+}(\hat{\Omega}) = \frac{1}{2}\frac{\hat{p}^i\hat{p}^j}{1+\hat{\Omega}\cdot\hat{p}}e^{+}_{ij}(\hat{\Omega}), \quad (3)$$

$$= \frac{1}{2}\frac{\hat{p}^i\hat{p}^j}{1+\hat{\Omega}\cdot\hat{p}}(\hat{m}_i\hat{m}_j - \hat{n}_i\hat{n}_j), \quad (4)$$

$$= \frac{1}{2}\frac{(\hat{p}^i\hat{m}_i)(\hat{p}^j\hat{m}_j) - (\hat{p}^i\hat{n}_i)(\hat{p}^j\hat{n}_j)}{1+\hat{\Omega}\cdot\hat{p}}. \quad (5)$$

Note that this is the definition of a dot product! i.e. $p^i\hat{m}_i = \hat{p}\cdot\hat{m}$. Therefore, the final (general) answer is:

$$F^{+} = \frac{1}{2}\frac{(\hat{p}\cdot\hat{m})^2 - (\hat{p}\cdot\hat{n})^2}{1+\hat{\Omega}\cdot\hat{p}}. \quad (6)$$

Similarly, for the cross polarization, we write

$$F^{\times}(\hat{\Omega}) = \frac{1}{2}\frac{\hat{p}^i\hat{p}^j}{1+\hat{\Omega}\cdot\hat{p}}e^{\times}_{ij}(\hat{\Omega}), \quad (7)$$

$$= \frac{1}{2}\frac{\hat{p}^i\hat{p}^j}{1+\hat{\Omega}\cdot\hat{p}}(\hat{m}_i\hat{n}_j + \hat{n}_i\hat{m}_j), \quad (8)$$

$$= \frac{1}{2}\frac{(\hat{p}^i\hat{m}_i)(\hat{p}^j\hat{n}_j) + (\hat{p}^i\hat{n}_i)(\hat{p}^j\hat{m}_j)}{1+\hat{\Omega}\cdot\hat{p}}, \quad (9)$$

(10)

and again using the definition of the dot product, one can write the final answer:

$$F^{\times}(\hat{\Omega}) = \frac{(\hat{p}\cdot\hat{m})(\hat{p}\cdot\hat{n})}{1+\hat{\Omega}\cdot\hat{p}}. \quad (11)$$

3. Give expressions for F^{+}_a, $F^{\times}_a$, F^{+}_b, $F^{\times}_b$ in your chosen coordinate frame.

   [Solution] In the reference frame given in Fig 1, one can now write down the antenna beam patterns as follows:

$$F^{\times}_a = 0, \quad (12a)$$

$$F^{+}_a = -\frac{1}{2}(1-\cos\theta), \quad (12b)$$

$$F^{\times}_b = \frac{(\sin\phi\ \sin\zeta)(\cos\theta\sin\zeta\cos\phi - \sin\theta\cos\zeta)}{1+\cos\theta\cos\zeta + \sin\theta\sin\zeta\cos\phi}, \quad (12c)$$

$$F^{+}_b = \frac{1}{2}\frac{(\sin\phi\sin\zeta)^2 - (\sin\zeta\cos\theta\cos\phi - \sin\theta\cos\zeta)^2}{1+\cos\theta\cos\zeta + \sin\theta\sin\zeta\cos\phi}. \quad (12d)$$

   This choice of geometry is commonly refereed to as the "computational frame", since here $F^{\times}_a = 0$, greatly simplifying analytical calculations to follow.

4. The overlap reduction function is given by integrating the antenna beam patterns over the sky, i.e. over all possible GW directions $\hat{\Omega}$:

$$^{(ab)}\Gamma(\zeta) = \int_0^{\pi}\sin\theta d\theta\int_0^{2\pi}d\phi\sum_A F^A_a(\hat{\Omega})F^A_b(\hat{\Omega}), \quad (13)$$

   where $\zeta \in [0,\pi]$ is the angle between the pulsars. Using your expressions for the antenna beam pattern, you may either integrate the above expression analytically (hint, use contour integration) or open the ipython notebook and do this numerically. Plot the result for all ζ. The result is the Hellings and Downs curve!

[Analytical Solution]

Substituting Eq. (12) into Eq. (13), the overlap reduction functions becomes:

$$ {}^{(ab)}\Gamma(\zeta) = -\frac{1}{4}\int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\phi \frac{(1-\cos\theta)(\sin^2\zeta\sin^2\phi - \sin^2\zeta\cos^2\theta\cos^2\phi - \cos^2\zeta\sin^2\theta + 2\sin\zeta\cos\zeta\sin\theta\cos\theta\cos\phi)}{1+\sin\zeta\sin\theta\cos\phi+\cos\zeta\cos\theta} \,. \tag{14} $$

One can write Eq (14) as the sum of two integrals:

$$ {}^{(ab)}\Gamma = \frac{1}{4}(Q+R)\,, \tag{15} $$

where

$$ Q = \int_0^{\pi} d\theta \sin\theta(1-\cos\theta) \int_0^{2\pi} d\phi(1-\cos\zeta\cos\theta - \sin\zeta\sin\theta\cos\phi) = \sqrt{4\pi}\left(1+\frac{\cos\zeta}{3}\right)\,, \tag{16} $$

and

$$ R = -2\sin^2\zeta \int_0^{\pi} d\theta \sin\theta(1-\cos\theta) I \tag{17} $$

$$ I \equiv \int_0^{2\pi} d\phi \frac{\sin^2\phi}{1+\cos\zeta\cos\theta+\sin\zeta\sin\theta\cos\phi}\,. \tag{18} $$

Eq (18) is evaluated via contour integration (or via Maple/Mathematica, whatever you like!)

$$ I = 2\pi \frac{1+\cos\zeta\cos\theta - |\cos\zeta+\cos\theta|}{\sin^2\zeta\sin^2\theta}\,, \tag{19} $$

$$ I = 2\pi \begin{cases} \left(\frac{1-\cos\zeta}{\sin^2\zeta}\right)\left(\frac{1-\cos\theta}{\sin^2\theta}\right), & 0<\theta<\pi-\zeta \\ \left(\frac{1+\cos\zeta}{\sin^2\zeta}\right)\left(\frac{1+\cos\theta}{\sin^2\theta}\right), & \pi-\zeta<\theta<\pi\,. \end{cases} \tag{20} $$

We can now write down the final form of R:

$$ R = -\frac{4\pi(1-\cos\zeta)}{\sqrt{4\pi}} \int_0^{\pi-\zeta} d\theta \frac{(1-\cos\theta)^2}{\sin\theta} - \frac{4\pi(1+\cos\zeta)}{\sqrt{4\pi}} \int_{\pi-\zeta}^{\pi} d\theta \sin\theta \tag{21} $$

$$ = \sqrt{4\pi}(1-\cos\zeta)4\ln\left(\sin\frac{\zeta}{2}\right)\,. \tag{22} $$

Using Eq (15), one may write the isotropic solution to Eq (14):

$$ {}^{(ab)}\Gamma(\zeta) = \frac{\sqrt{\pi}}{2}\left[1+\frac{\cos\zeta}{3}+4(1-\cos\zeta)\ln\left(\sin\frac{\zeta}{2}\right)\right]\,. \tag{23} $$

This equation is the Hellings and Downs curve up to a multiplicative factor. If you're interested in knowing more about this, see [3].

5. Which of these steps implicitly assumes an isotropic background? Where would we modify this calculation to take in to account anisotropy in the GW background? More on this in [4].

   [Solution] We can decompose the GW power on the sky by using a basis of spherical harmonics. Mathematically, we write this as $P(\hat{\Omega}) = \sum_{lm} c_{lm} Y_{lm}$, where the Y_{lm} are the regular spherical harmonics. The dependence on $P(\hat{\Omega})$ is added to the evaluation of the overlap reduction reduction function before integration, thus allowing for any angular power distribution.

$$ {}^{(ab)}\Gamma(\zeta) = \int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} d\phi \sum_A F_a^A(\hat{\Omega}) F_b^A(\hat{\Omega}) P(\hat{\Omega})\,, \tag{24} $$

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[3] C. M. F. Mingarelli and T. Sidery, Phys. Rev. D **90**, 062011 (2014).
[4] C. M. F. Mingarelli, T. Sidery, I. Mandel, and A. Vecchio, Phys. Rev. D **88**, 062005 (2013), 1306.5394.