1. *Pauli exclusion.* Consider population of N fermions of mass m_f confined in a one-dimensional "box." Each fermion must be in a different quantum state, but two fermions can have the same momentum $m_f v$, where v is velocity, if they have opposing spins. The momentum states have values seperated by $\Delta(m_f v)$, and the values can be positive or negative in the single dimension. The fermions will fill momentum states up to a maximum magnitude $(N/4)\Delta(m_f v)$.

   Suppose the electrons are confined to a three-dimensional box. Argue that, to within factors of order unity, the maximum momentum is $N^{1/3}\Delta(m_f v)$.

2. *Heisenberg uncertainty.* N fermions are confined to a star of radius R; this sets the uncertainty in their position in each direction to be at least $\Delta x = 2R$. The Heisenberg uncertainty principle, $\Delta x \Delta(m_f v) \geq \hbar/2$, implies a corresponding uncertainty in momentum which ensures a maximum momentum value as seen in the previous part. What is the largest momentum of a fermion in the star? The average momentum is of this order. *(Answer: $N^{1/3}\hbar/R$)*

3. You may recall that pressure is provided by particles moving randomly in the ideal gas law, $pV = NkT$. Temperature kT can be replaced by the expression for average kinetic energy $2\langle KE\rangle/3$, in which case the law also applies to the pressure provided by degenerate particles:

$$p = \frac{2}{3}\frac{N}{V}\langle KE\rangle \tag{1}$$

   Momentum $m_f v$ in each direction contributes a kinetic energy of $(m_f v)^2/2m_f$. What is the order-of-magnitude pressure provided by the average motion of N degenerate fermions in a star of radius R? Note that $V \sim R^3$. *(Answer: $\frac{\hbar^2 N_f^{5/3}}{m_f R^5}$)*

4. Stellar size

   Consider the Newtonian equation of hydrostatic equilibrium, where ρ is density, p is pressure, and m is the mass enclosed by a surface of radius r:

$$\frac{dp(r)}{dr} = -\frac{Gm(r)\rho(r)}{r^2} \tag{2}$$

$$\frac{dm(r)}{dr} = 4\pi r^2 \rho(r) \tag{3}$$

   We can estimate relationships between characteristic quantities by approximating the equilibrium equations as

$$\frac{p_c}{R} \sim \frac{GM\langle\rho\rangle}{R^2} \tag{4}$$

$$M \sim \frac{4}{3}\pi R^3 \langle\rho\rangle \tag{5}$$

   where R is the radius of the star, M the total mass, p_c the central pressure, and angle brackets denote averages over the star.

   Combine the approximate equilibrium equations to relate central pressure p_c to the mass M and radius R of the star. *(Answer: $p = 3GM^2/4\pi R^4$)*

5. The total mass of the star will be dominated by the number of baryons N_b of mass m_b. For a neutron star, assume degenerate neutrons provide both mass and pressure: $M = m_n N_n$, and $p = \frac{\hbar^2 N_n^{5/3}}{4 m_n R^5}$ What is the radius of the star, in terms of mass? What is the radius of a 1.4 solar mass neutron star? *(Answer:* $R = \hbar^2 (M/m_n)^{5/3}/(G m_n M^2)$*,* 6 km)

   The true radius of a neutron star is closer to 12 km ±2 km or so; our calculation neglected many constants, and there is additional pressure provided by the strong interaction at nuclear density and above.

6. Using our approximate relationship for pressure, a mass of $1.4 M_\odot$, and a radius of 12 km, estimate the pressure at the center of a neutron star. ($p = 3GM^2/4\pi R^4$, 10^{33} pascals).

7. Magnetic pressure is an energy density associated with magnetic fields, given by $B^2/(2\mu_0)$ (in vacuum). What magnetic field gives pressure comparable to the pressure at the core of a neutron star? (10^{14} Tesla or 10^{18} Gauss).

8. Go back to the original form of the ideal gas law, with number of particles given by $N_n = M/m_n$. What temperature would give pressure comparable to the pressure at the core of a neutron star? (10^{11} Kelvin)