Numerical relativity exercise: solutions

1. **Wave equation in 1st order form**

   (a) We wish to show

$$\partial_t\psi = -\Pi, \tag{1}$$
$$\partial_t\Pi + \partial_x\Phi = 0, \tag{2}$$
$$\partial_t\Phi + \partial_x\Pi = 0, \tag{3}$$

   where

$$\Pi \equiv -\partial_t\psi, \tag{4}$$
$$\Phi \equiv \partial_x\psi. \tag{5}$$

   First the easy part: Eq. (1) is the same as the definition (4).
   Next, take the time derivative of Eq. (4), which gives

$$\partial_t\Pi = -\partial_t\partial_t\psi. \tag{6}$$

   Replace the right-hand side of Eq. (6) with the wave equation in its usual form $\partial_t\partial_t\psi = \partial_x\partial_x\psi$ to obtain

$$\partial_t\Pi = -\partial_x\partial_x\psi. \tag{7}$$

   Now combine Eq.
   Now take the x derivative of Eq. (5). The result is

$$\partial_x\Phi = \partial_x\partial_x\psi. \tag{8}$$

   Add Eqs. (7) and (8) to obtain the desired result, Eq. (2)
   Finally, take the time derivative of Eq. (5) and add to it the x derivative of Eq. (4). Using the commutivity of partial derivatives, this yields Eq. (3).

2. **Finite difference operator**

   (a) See attached python code.

   (b) See attached python code.

   (c) See and run attached python code. The errors are proportional to $1/N^2$, as shown by the blue curve on the left of Fig. 1. This is why the particular finite-difference method we are using is 2nd order. For an pth order finite difference method, errors *for smooth functions* should scale like $1/N^p$.

   (d) See and run attached python code. For non-smooth functions, one does not obtain the same error behavior as for smooth functions. In this case, the error in the derivative scales only like $1/N$, 1st order, as shown by the green curve on the left of Fig. 1. For problems with shocks and other discontinuities, going to higher order generally does not produce higher accuracy.

3. **Spectral representation:**

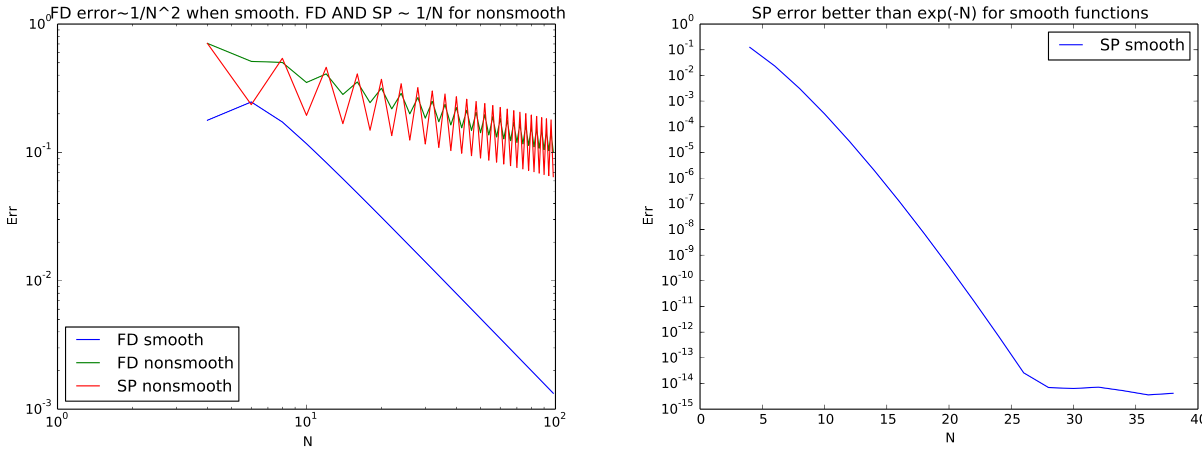

Figure 1: Derivative errors as a function of N. The left is on a log-log plot, and shows finite-difference errors for a smooth function, and both finite-difference and spectral errors for a nonsmooth function. the right is on a semilog scale and shows spectral errors for a smooth function.

(a) See attached python code.

(b) See and run attached python code. Here the error decreases faster than exponentially, as shown by the curve on the right panel of Fig. 1. In general, for smooth functions we expect the error to scale like e^{-N} for spectral methods, thus providing much smaller errors than finite difference methods. Note that at large enough N, the error no longer decreases with increasing N. This is because computers use finite-precision arithmetic: floating-point numbers are only represented to about 16 digits. The error associated with finite-precision arithmetic is typically called "roundoff error".

(c) See and run attached python code. Here the error still decreases like $1/N$, first order, 1st order, as shown by the red curve on the left of Fig. 1. The moral of the story is that although spectral methods may be superior for smooth solutions, they generally do not provide higher than first order accuracy for nonsmooth solutions.

4. **Evolving the wave equation**

(a) See attached python code.

(b) See attached python code.

(c) See attached python code.

(d) See and run attached python code, and Figure 2.

(e) See attached python code.

(f) See attached python code, and Fig. 3 We find from trial and error that the CFL limit is about 2.8. One can derive the actual value of the CFL limit for 4th order Runge-Kutta and a wave equation; the answer is $2\sqrt{2}$.

(g) See attached python code and Fig. 4. We find that the CFL limit is roughly 0.94 for spectral methods.

(h) See attached python code. To keep $\psi(x,t)$ above -10^{-2}, we need about 144 point for 2nd order finite-differencing, and bout 32 points for Fourier spectral methods.

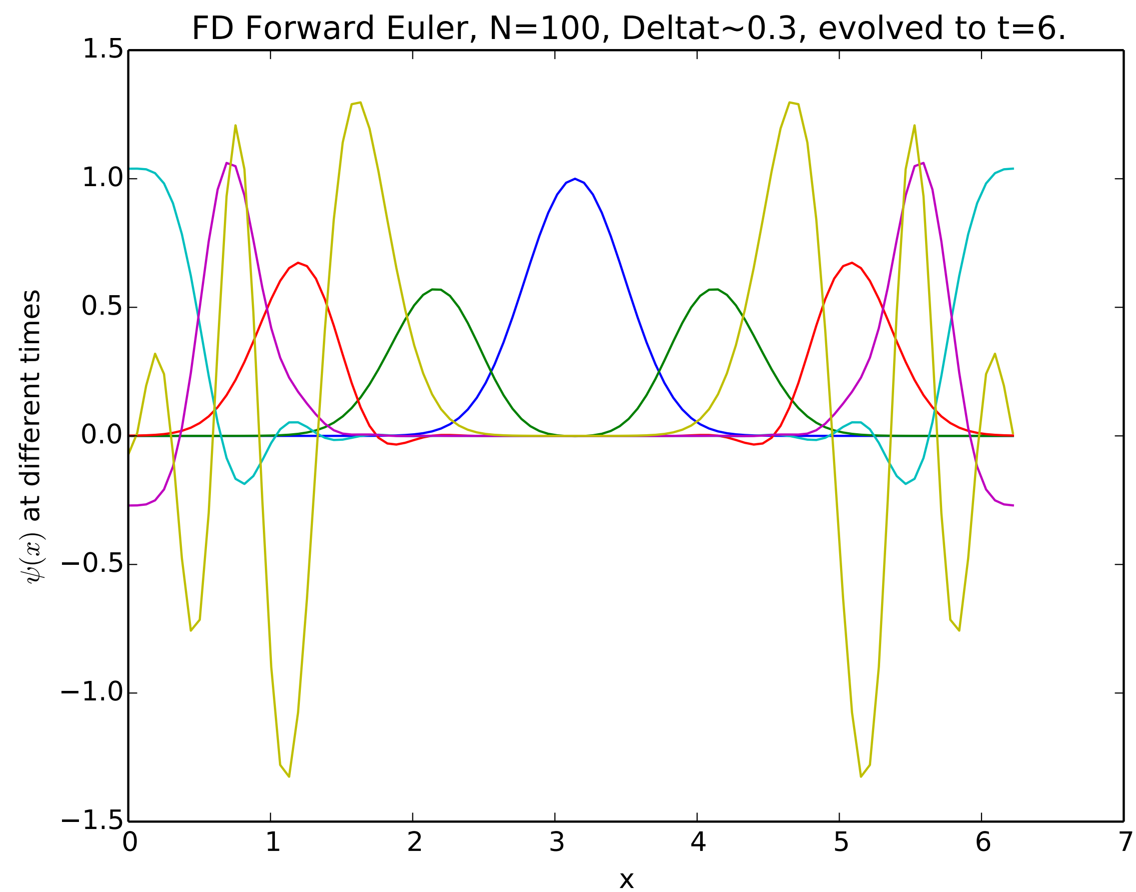

Figure 2: Left: Numerical solution of the wave equation plotted at $t = 1$, $t = 2$, ... through $t = 6$.

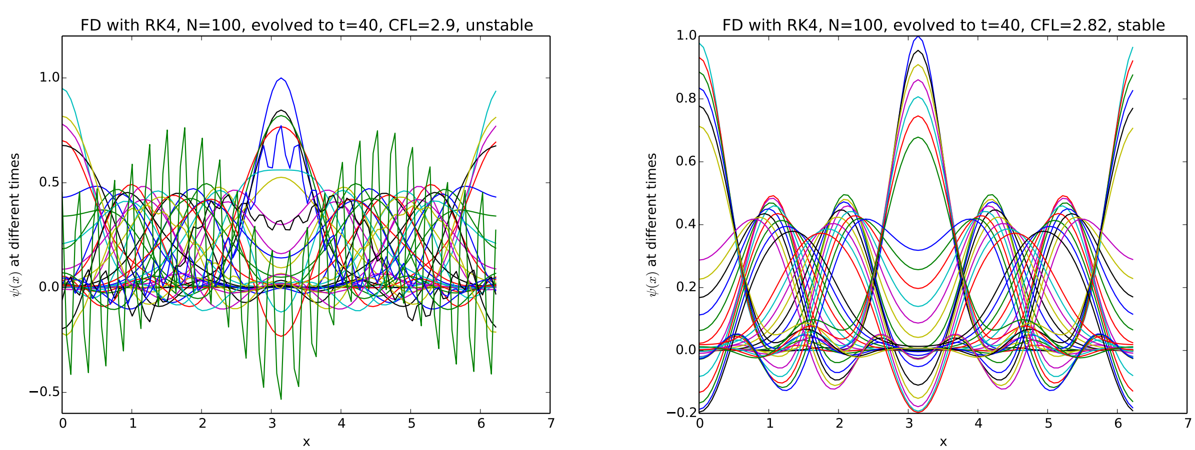

Figure 3: Left: Numerical solution of the wave equation plotted at $t = 1$, $t = 2$, ... through $t = 40$ for 2nd-order finite-differencing. The CFL number is 2.9, and the evolution is unstable. Right: Same except the CFL number is 2.82, and the evolution is stable.

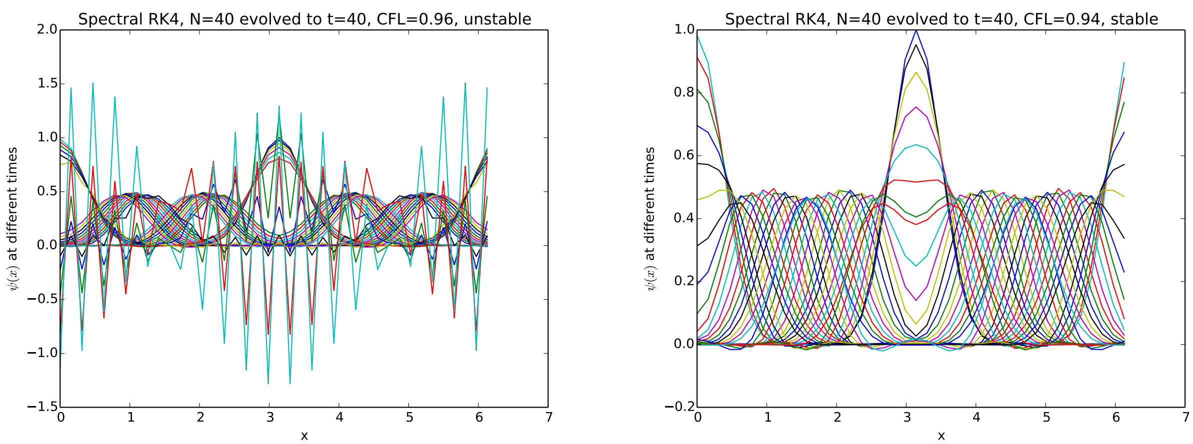

Figure 4: Left: Numerical solution of the wave equation plotted at $t = 1$, $t = 2$, ... through $t = 40$ for Fourier spectral methods. The CFL number is 0.96, and the evolution is unstable. Right: Same except the CFL number is 0.94, and the evolution is stable.