Friday practicum: stochastic gravitational wave backgrounds

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I. INTRODUCTION

In today's lectures, you have explored Testing General Relativity with GW observations, even do some cosmology with them, and how to detect low frequency GWs using Pulsar Timing Arrays. The following exercises are meant to cement these concepts. These come in at least 2 flavors: general exercises, and advanced ones for those who may already be familiar with these concepts and are up for a challenge!

II. EXERCISES

Do as many of the following exercises as time allows. If you get stuck, please ask a Lecturer or a teaching assistant for help.

A. Pulsar Timing Arrays

Pulsar timing arrays are sensitive to GWs in the nanoHertz frequency regime. Sources here include supermassive black hole binaries, primordial (or relic) GWs from inflation, and cosmic strings. The goal of this exercise is to derive the overlap reduction function for an isotropic stochastic GW background. This is often referred to as the Hellings and Downs curve [1].

1. For a pair of pulsars a and b, define a reference frame. Specifically, write down the unit vectors which point to the pulsars, $\hat{p}_a$, $\hat{p}_b$, the direction of GW propagation $\hat{\Omega}$ and the principal axes $\hat{m}$, $\hat{n}$, such that $\hat{m} \times \hat{n} = \hat{\Omega}$. *Hint, place pulsar a on the z-axis and b in the x-z plane*

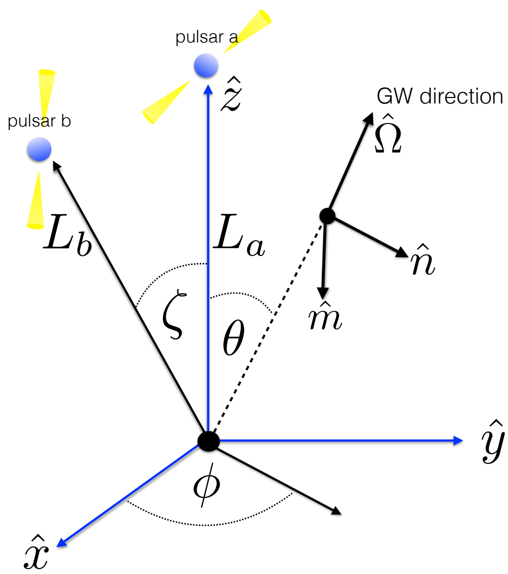

FIG. 1: Suggested geometry: pulsar a is on the z-axis at a distance L_a from the origin (solar system barycentre), pulsar b is in the x-z plane at a distance L_b from the origin making an angle ζ with pulsar a. $\hat{\Omega}$ is the direction of GW propagation with principal axes $\hat{m}$ and $\hat{n}$ such that $\hat{m} \times \hat{n} = \hat{\Omega}$. The polar and azimuthal angles are given by θ and ϕ, respectively.

2. Write down the general expression for the antenna beam pattern,

$$F^A(\hat{\Omega}) = \frac{1}{2}\frac{\hat{p}^i\hat{p}^j}{1+\hat{\Omega}\cdot\hat{p}}e^A_{ij}(\hat{\Omega})\,, \tag{1}$$

where $A = +, \times$ is the GW polarization. The polarization tensors $e^+_{ij}(\hat{\Omega}) = \hat{m}_i\hat{m}_j - \hat{n}_i\hat{n}_j$, $e^\times_{ij}(\hat{\Omega}) = \hat{m}_i\hat{n}_j + \hat{n}_i\hat{m}_j$. For further reading, see [2, 3].

3. Give expressions for F^+_a, $F^\times_a$, F^+_b, $F^\times_b$ in your chosen coordinate frame

4. The overlap reduction function is given by integrating the antenna beam patterns over the sky, i.e. over all possible GW directions $\hat{\Omega}$:

$$^{(ab)}\Gamma(\zeta) = \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi \sum_A F^A_a(\hat{\Omega})F^A_b(\hat{\Omega})\,, \tag{2}$$

where $\zeta \in [0,\pi]$ is the angle between the pulsars. Using your expressions for the antenna beam pattern, you may either integrate the above expression analytically (hint, use contour integration) or open the ipython notebook and do this numerically. Plot the result for all ζ. The result is the Hellings and Downs curve!

5. Which of these steps implicitly assumes an isotropic background? Where would we modify this calculation to take in to account anisotropy in the GW background? More on this in [4].

B. Testing GR

Having built the above tools, Let us move on to explore testing GR in the context of GW background in PTAs. We will do this in two steps: first, we shall compute the overlap reduction function in more general theories of gravity.

6. Suppose GW has a scalar component, i.e., for a gravitational wave propagating along the $\hat{\Omega}$ direction, we have

$$e^\circ_{ij}(\Omega) = \hat{m}_i\hat{m}_j + \hat{n}_i\hat{n}_j\,. \tag{3}$$

Write down the antenna beam pattern and compute the overlap reduction function.

7. Suppose, *instead*, that the graviton is actually massive, and that the GW has a dispersion relation

$$\omega^2 = m_g^2 + k^2\,. \tag{4}$$

In other words, a single-frequency plane GW produces the following metric perturbation:

$$h_{ij}(t,\mathbf{x}) = e^{-i\omega(k)t+ik\hat{\Omega}\cdot\mathbf{x}}\left[H_+e^+_{ij} + H_\times e^\times_{ij}\right] \tag{5}$$

Write down the antenna beam pattern and the overlap reduction function.

As a second step, let us explore how to distinguish between a standard GR background and a non-GR background.

8. Based on our result of the overlap reduction function for a scalar background, can we independently extract the scalar and the tensor background?

9. In the massive gravity case: if we do detect a stochastic background, can we use it to constrain the graviton mass? How well can we do so?

C. Cosmology with GWs

11. Solve the scalar-wave propagation equation $\Box\phi(x) = 0$ in the Friedmann–Robertson–Walker spacetime

$$ds^2 = -dt^2 + a(t)^2[dx^2 + dy^2 + dz^2], \tag{6}$$

and show that it admits solutions in the form

$$\phi(r,t) = \frac{1}{r\,a(t_0)}g(t - r/c), \tag{7}$$

where $r = \sqrt{x^2+y^2+z^2}$ and t_0 is the time at the present epoch. [Hint: introduce a conformal time η that satisfies $d\eta = dt/a(t)$, and use the curved-space d'Alembertian $\Box = 1/\sqrt{-g}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu)$].

12. Solve problem 19.10 from Lightman et al.'s *Problem book in relativity and gravitation* (Princeton, 1975; online at `http://www.nrbook.com/relativity`).

> *Problem 19.10.* Suppose astronomers are able to find a family of objects whose absolute luminosities L are known. Suppose their apparent luminosities ℓ (or equivalently their luminosity distance d_L) and their red-shift z are measured. Using the Robertson-Walker line element, find an expression for ℓ (or d_L) as a function of L, z, H_0 and q_0 for small z.

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[3] M. Anholm, S. Ballmer, J. D. E. Creighton, L. R. Price, and X. Siemens, Phys. Rev. D **79**, 084030 (2009), 0809.0701.
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