Detection of Compact Binaries

The Stationary Phase Approximation

In a previous problem set, you computed the gravitational wave's amplitude and phase as a function of time; these are the physical quantities measured by the LIGO detectors. In practice, both searches and parameter estimation often make use of the Fourier transform of $h(t)$, given by

$$\tilde{h}(f) = \int_{-\infty}^{\infty} h(t) e^{2\pi i f t}\, dt \tag{1}$$

Question 1. What are the units of $\tilde{h}(f)$?

Question 2. Show that $\tilde{h}^*(f) = \tilde{h}(-f)$ where the star denotes complex conjugation. Hint: remember that the gravitational wave signal $h(t)$ is purely real (i.e. it never has an imaginary component).

Question 3. We can use the stationary phase approximation to derive an expression for $\tilde{h}(f)$ directly in the frequency domain. First, we write the gravitational wave signal as

$$h(t) = A(t)\cos\left[\phi(t)\right] = A(t)\left[e^{i\phi(t)} + e^{-i\phi(t)}\right]. \tag{2}$$

The nature of the inspiral signal tells us that

$$\dot{\phi}(t) > 0 \text{ and } \dot{\phi}(t) \gg \frac{\dot{A}(t)}{|A(t)|} \tag{3}$$

that is, the frequency of the waves is much higher than the rate of change in the frequency or the amplitude. Use Eqs. (1) and (2) to write the Fourier transform of the gravitational wave. You should obtain an integral of the form

$$\tilde{h}(f) = \int_{-\infty}^{\infty} A(t)\left(e^{A} + e^{B}\right)\, dt \tag{4}$$

where A and B are two phase terms.

By evaluating the Fourier transform using the stationary phase approximation[1], show that

$$\tilde{h}(f) = B(f) e^{i\Psi(f) - i\pi/4} \tag{5}$$

where the amplitude of the Fourier transform is

$$B(f) = \frac{1}{2}\sqrt{\frac{dt(f)}{df}}A(t(f)) \propto f^{-7/6} \tag{6}$$

and the phase of the Fourier transform is, expressed as a function of frequency f is

$$\Psi(f) = 2\pi f t(f) - \phi(f). \tag{7}$$

Question 4. The waves' phase

$$\phi(t) = \int 2\pi f dt \tag{8}$$

can be thought of equally well as a function of time t, or a function of frequency f that is reached at time t. or as a function of $v = (\pi M f)^{1/3}$. Show that

$$\phi(f) = \phi_{\rm ref} + 2\int_{v}^{v_{\rm ref}} \frac{v'^3 dE(v')/dv'}{\mathcal{F}(v')} dv' \tag{9}$$

where $\phi_{\rm ref}$ is the value of the phase when the reference frequency is reached, and the v's on the right hand side are to be though of as functions of f, $v = (\pi M f)^{1/3}$. Derive a formula for $\phi(f)$ as a post-Newtonian expansion in v accurate to 1.5 post-Newtonian order.

[1] http://en.wikipedia.org/wiki/Stationary_phase_approximation

Question 5. Use your post-Newtonian expansions for $t(f) \equiv t_f$ and $\phi(f)$ to obtain the following expansion for the waves' phase

$$\Psi(f) = 2\pi f t_{\rm ref} - \phi_{\rm ref} + \frac{3}{128}(\pi \mathcal{M} f)^{-5/3}\left[1 + \left(\frac{3715}{756} + \frac{55}{9}\eta\right)v^2 - 16\pi v^3 + \mathcal{O}(v^4)\right]. \tag{10}$$

Hint, if you need it:

$$2\pi f t_f = 2\pi f t_{\rm ref} + \frac{5}{128\mu}v^3\left[\left(\frac{1}{v_{\rm ref}^8} - \frac{1}{v^8}\right) + \frac{743}{252}\left(\frac{1}{v_{\rm ref}^6} - \frac{1}{v^6}\right) - \frac{32\pi}{5}\left(\frac{1}{v_{\rm ref}^5} - \frac{1}{v^5}\right)\right] + \cdots \tag{11}$$

and

$$\phi(f) = \phi_{\rm ref} + \frac{M}{16\mu}\left[\left(\frac{1}{v_{\rm ref}^5} - \frac{1}{v^5}\right) + \frac{3715}{1008}\left(\frac{1}{v_{\rm ref}^3} - \frac{1}{v^3}\right) - 10\pi\left(\frac{1}{v_{\rm ref}^2} - \frac{1}{v^2}\right)\right] + \cdots \tag{12}$$

Derivation of the Matched Filter

In this problem, we will consider the construction of the likelihood ratio Λ for the detector data $s(t)$ and the gravitational wave signal $h(t)$. Assume that the noise is stationary and Gaussian with zero mean value

$$\langle n(t) \rangle = 0 \tag{13}$$

where angle brackets denote averaging over different ensembles of the noise. The (one sided) power spectral density $S_n(|f|)$ of the noise is defined by

$$\langle \tilde{n}(f)\tilde{n}(f')\rangle = \frac{1}{2}S_n(|f|)\delta(f - f') \tag{14}$$

where $\tilde{n}(f)$ is the Fourier transform of $n(t)$. The probability density of obtaining a particular instantiation of detector noise is

$$p(n) = \mathcal{K}\exp\left[-\frac{1}{2}(n|n)\right] \tag{15}$$

where $\mathcal{K}$ is a normalization constant and the inner product $(\cdot|\cdot)$ is given by

$$(a \mid b) \equiv \int_{-\infty}^{\infty} df\, \frac{\tilde{a}^*(f)\tilde{b}(f) + \tilde{a}(f)\tilde{b}^*(f)}{S_n(|f|)}. \tag{16}$$

Question 1. What is the probability density of obtaining the interferometer output $s(t)$ in the absence of signal, i.e. $s(t) = n(t)$? Denote this as $p(s|0)$.

Question 2. What is the probability density of obtaining $s(t)$ in the presence of a signal, i.e. when $s(t) = n(t) + h(t)$? Denote this as $p(s|h)$.

Question 3. Given these two quantities, calculate the likelihood ratio Λ and show that it is a monotonically increasing function of $(s|h)$.

Question 4. Show that since both the interferometer output and the template are real functions of time, the inner product in equation (16) becomes

$$(a \mid b) = 2\int_{-\infty}^{\infty} df\, \frac{\tilde{a}(f)\tilde{b}^*(f)}{S_n(|f|)}. \tag{17}$$

Question 5. Calculate the mean of $(s|h)$ if the detector data contains noise alone. That is, compute $\langle(s|h)\rangle$ over an ensemble of detector outputs.

Question 6. Calculate the expectation value of $(s|h)$ if the detector data contains noise alone. That is, compute $\langle(s|h)^2\rangle$ over an ensemble of detector outputs. Hint: use the definition of the one-sided power spectral density given in Eq. (14).

Question 7. Recall from the lecture that since we need to maximize over the phase of the gravitational waveform, the likelihood used in practice is a monotonically increasing function of

$$|z| = \sqrt{(s|h_c)^2 + (s|h_s)^2}. \tag{18}$$

where h_c and h_s are the two orthogonal phases of the gravitational waveform. The normalized single detector signal-to-noise ratio used by LIGO defined by

$$\rho^2 = \frac{|z|^2}{\sigma^2} \tag{19}$$

where $\sigma^2 = (h_c|h_c) = (h_s|h_s)$. Calculate $\langle\rho^2\rangle$ when the detector output contains noise alone. What is the statistical distribution of ρ^2 and why?

Question 8. If a gravitational wave signal is present, then its location in time is defined by the *end time* parameter t_e of the waveform. In the above discussion of the optimal receiver, we implicitly knew the location of the signal in the data to have $t_e = 0$. Now suppose that the inspiral waveform ends at some unknown time t_e. We may write the signal we are searching for as $h(t' - t_e)$. By considering the Fourier transform of this signal

$$\int_{-\infty}^{\infty} e^{-2\pi i f t'} h(t' - t_e)\, dt' \tag{20}$$

show that the signal-to-noise ratio for a chirp that ends at time t is

$$\rho(t) = \frac{1}{\sigma}\sqrt{(s|h_c(t))^2 + (s|h_s(t))^2} \tag{21}$$

where the quantities $(s|h_c(t))$ and $(s|h_s(t))$ can be obtained by inverse Fourier transforms of the form

$$(s|h_c(t_e)) = 2\int_{-\infty}^{\infty} df e^{2\pi i f t_e} \frac{\tilde{s}(f)\tilde{h}_c^*(f)}{S_n(|f|)}. \tag{22}$$

Question 9. In LIGO analysis, we choose the template $\tilde{h}_c(f)$ to be at a canonical distance of 1 Mpc. The effective distance $\mathcal{D}$ to a chirp detected with signal to noise ratio ρ^2 is then given by

$$\mathcal{D} = \frac{\sigma}{\rho}\,\mathrm{Mpc}. \tag{23}$$

Verify that the units in this equation are consistent with Eq. (22). Obviously, as ρ increases $\mathcal{D}$ decreases and as σ increases, $\mathcal{D}$ increases. Is this consistent with your physical intuition and why? What do the quantities ρ and σ measure?