Stationary Phase Approximation

1) Since dt has units of time, $\tilde{h}(f)$ has units of time or Hz^{-1}. We often define a "unit" of strain, even though strain is dimensionless, so $\tilde{h}(f)$ is said to have units strain / Hz.

2) $$\tilde{h}(f) = \int_{-\infty}^{\infty} e^{2\pi i f t}\, h(t)\, dt$$

$$\Rightarrow \quad \tilde{h}^*(f) = \int_{-\infty}^{\infty} e^{-2\pi i f t}\, h^*(t)\, dt$$

But since $h(t)$ is real, $h^*(t) = h(t)$, so

$$\tilde{h}^*(f) = \int_{-\infty}^{\infty} e^{2\pi i (-f) t}\, h(t)\, dt = \tilde{h}(-f)$$

3) Recall that $h(t) = A(t) \cos \phi(t)$, so

$$\tilde{h}(f) = \int_{-\infty}^{\infty} e^{2\pi i f t} A(t) \cos \phi(t)\, dt$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} e^{2\pi i f t} A(t) \left[e^{i\phi(t)} + e^{-i\phi(t)} \right] dt$$

Now consider the contribution to the integral from these two terms:

$e^{-i\phi(t) + 2\pi i f t}$ is always oscillating fast, except in near the stationary phase point t_f where $\dot{\phi} = 2\pi f$

$e^{i\phi(t) + 2\pi i f t}$ is always oscillating fast, so won't contribute

(Note we have restricted to $f > 0$, given the result of question 2 above)

Now expand around t_f:

$$\phi(t) = \phi(t_f) + \dot\phi(t_f)(t-t_f) + \tfrac{1}{2}\ddot\phi(t_f)(t-t_f)^2 + \dots$$

$$= \phi(t_f) + 2\pi f\,(t-t_f) + \pi \dot f(t_f)\,(t-t_f)^2 + \dots$$

Substituting this back into the integral

$$\tilde h(f) \approx \frac{1}{2}\int_{-\infty}^{\infty} e^{2\pi i f t} A(t)\, e^{-i\phi(t)}\, dt$$

$$\approx \frac{1}{2} A(t_f)\, e^{-i\phi(t_f) + 2\pi i f t_f} \underbrace{\int_{-\infty}^{\infty} e^{-i\pi \dot f(t_f)(t-t_f)^2}\, dt}_{= e^{-i\pi/4} / \sqrt{\dot f(t_f)}}$$

$$= \underbrace{\frac{1}{2}\sqrt{dt_f/df}\; A(t_f)}_{B(f)}\; e^{i\underbrace{[2\pi f t_f - \phi(t_f)]}_{\Psi(f)} - i\pi/4}$$

$$= B(f)\, e^{i\Psi(f) - i\pi/4}$$

Now $A(t_f) = f^{2/3}$

and $\dfrac{dt_f}{df} \sim \dfrac{1}{f^{11/3}}$ at leading order

so $B(f) \propto f^{-7/6}$

4) $$\phi = \phi_{ref} + \int_{t_{ref}}^{t} 2\pi f \, dt$$

$$= \phi_{ref} + \int_{v_{ref}}^{v} 2\pi f' \, \frac{1}{\dot{v}'} \, dv'$$

$$= \phi_{ref} + 2 \int_{v_{ref}}^{v} \left(\frac{v'^3}{M} \right) \left(\frac{dE(v')/dv'}{-\mathcal{F}(v')} \right) dv'$$

$$= \phi_{ref} + 2 \int_{v}^{v_{ref}} \frac{v'^3 \, dE/dv'}{M \mathcal{F}(v')} \, dv'$$

Now $$\frac{v^3 \, dE/dv}{\mathcal{F}(v)} = -\frac{5M^2}{32\mu v^6} \left(1 + \frac{743}{336} v^2 - 4\pi v^3 + \mathcal{O}(v^4) \right)$$

$$\Rightarrow \quad \phi = \phi_{ref} + 2 \int_{v}^{v_{ref}} \frac{v'^3 \, dE(v')/dv'}{M \mathcal{F}(v')} \, dv'$$

$$= \phi_{ref} + \frac{M}{16\mu} \left[\left(\frac{1}{v_{ref}^5} - \frac{1}{v^5} \right) + \frac{3715}{1008} \left(\frac{1}{v_{ref}^3} - \frac{1}{v^3} \right) - 10\pi \left(\frac{1}{v_{ref}^2} - \frac{1}{v^2} \right) + \dots \right]$$

5) We need to compute

$$2\pi f t_f - \phi(f)$$

$$= 2\pi f t_{ref} - \phi_{ref} + \frac{3M}{128\mu v^5}\left[1 + \frac{3715}{756}v^2 - 16\pi v^3 + \mathcal{O}(v^4)\right]$$

$$+ \frac{5M}{128\mu} v^3 \left[\frac{1}{V_{ref}^8} + \frac{743}{252}\frac{1}{V_{ref}^6} - \frac{32\pi}{5}\frac{1}{V_{ref}^5}\right]$$

$$+ \frac{M}{16\mu}\left[\frac{1}{V_{ref}^5} + \frac{3715}{1008}\frac{1}{V_{ref}^3} - 10\pi\frac{1}{V_{ref}^2}\right]$$

Let us set $V_{ref} = +\infty$. We can do this in PN, as our expressions are for point particles. Effectively, this means that t_{ref} is formally the time at which $f_{GW} \to \infty$. Then

$$\Psi(f) = 2\pi f t_{ref} - \phi_{ref} + \frac{3M}{128\mu}\frac{1}{(\pi M f)^{5/3}}\left[1 + \frac{3715}{756}V^2 - 16\pi v^3 + \mathcal{O}(v^4)\right]$$

$$= 2\pi f t_{ref} - \phi_{ref} + \frac{3}{128}\left(\pi \mathcal{M} f\right)^{-5/3}\left[1 + \frac{3715}{756}v^2 - 16\pi v^3 + \mathcal{O}(v^4)\right]$$

which is Eq. (13) up to the η term in the 1pN term, since our $\mathcal{F}$ and E did not include η. This expression is accurate in the limit $\eta \to 0$.

Notice that the leading order phasing depends only on the chirp mass $\mathcal{M}$.

Derivation of the Matched Filter

1) If $s(t) = n(t)$ then $p(s|0)$ is trivially

$$p(s|0) = K \exp\left[-\tfrac{1}{2}(s|s)\right]$$

2) If $s(t) = n(t) + h(t)$, then replace $n \rightarrow s-h$ in Eq (18) and

$$p(s|h) = K \exp\left[-\tfrac{1}{2}(s-h\,|\,s-h)\right]$$

3) $$\Lambda = \frac{p(s|h)}{p(s|0)} = \frac{\exp\left[-\frac{1}{2}(s-h|s-h)\right]}{\exp\left[-\frac{1}{2}(s|s)\right]}$$

$$= \exp\left[-\frac{1}{2}\left[(s|s) - 2(s|h) - (h|h)\right] + \frac{1}{2}(s|s)\right]$$

$$= \exp\left[(s|h) - \underbrace{\tfrac{1}{2}(h|h)}_{\text{constant}}\right]$$

So Λ is a monotonically increasing function of $(s|h)$.

4) This follows from the fact that $h(t)$ is real, so $h^*(f) = h(-f)$.

5) $\langle (s|h) \rangle = \langle (n|h) \rangle$

$$= \int_{-\infty}^{\infty} \frac{\langle \tilde{n}(f) \rangle \, \tilde{h}^*(f)}{S_n(|f|)} \, df$$

But since $\langle n(t) \rangle = 0$, $\langle \tilde{n}(f) \rangle = 0$, so

$\langle (s|h) \rangle = 0$.

6) $$\langle (s|h)^2 \rangle = 4 \left\langle \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} df \, df' \, \frac{\tilde{n}(f) \tilde{h}^*(f) \tilde{n}^*(f') \tilde{h}(f')}{S_n(f) S_n(f')} \, df \right\rangle$$

$$= 4 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} df \, df' \, \frac{\langle \tilde{n}(f) \tilde{n}^*(f') \rangle \tilde{h}^*(f) \tilde{h}(f')}{S_n(f) S_n(f')}$$

$$= 4 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} df \, df' \, \frac{\frac{1}{2} S_n(f) \, \delta(f-f') \, \tilde{h}^*(f) \tilde{h}(f')}{S_n(f) S_n(f')}$$

$$= 2 \int_{-\infty}^{\infty} df \, \frac{\tilde{h}(f) \tilde{h}^*(f)}{S_n(f)}$$

$$= (h|h)$$

7) Using the above

$$\langle \rho^2 \rangle = \frac{1}{\sigma^2} \langle (n|h_c)^2 + (n|h_s)^2 \rangle = \frac{(h|h) + (h|h)}{(h|h)} = 2$$

ρ^2 is chi-squared distributed with two degrees of freedom, since it is the sum of the squares of two Gaussian random variables when $s = n$.

8) $$\int_{-\infty}^{\infty} e^{-2\pi i f t} h(t' - t_e)\, dt = e^{-2\pi i f t_e} \int_{-\infty}^{\infty} e^{-2\pi i f t'} h(t')\, dt'$$

$$= e^{-2\pi i f t_e} \tilde{h}(f)$$

Therefore the matched filter snr for a waveform ending at t_e is

$$(s|h_c(t_e)) = 2 \int_{-\infty}^{\infty} df\, e^{2\pi i f t_e} \frac{\tilde{s}(f)\tilde{h}_c^*(f)}{S_n(f)}$$

and similarly for h_s.

9) If $h = h_{1Mpc}$, σ has units of Mpc and ρ is dimensionless. So D has units of Mpc

σ is a measure of the noise in the detector
ρ is a measure of signal strength relative to the noise

To get the same ρ in a more sensitive detector (bigger σ) the signal must be further away.

Similarly, if the detector sensitivity is constant, bigger ρ means closer signal.