CGWAS 2015: Hands-On Exercises

Sean Carroll

First two problems are to give practice with tensors; next two are more physical. Newton's constant: $G = 6.67 \times 10^{-8}\,\mathrm{cm}^3\,\mathrm{g}^{-1}\,\mathrm{s}^{-2}$. Speed of light: $c = 3.00 \times 10^{10}\,\mathrm{cm}\,\mathrm{s}^{-1}$. Mass of the Sun: $M_\odot = 2 \times 10^{33}\,\mathrm{g}$.

1. Given a vector n^μ normalized to $g_{\mu\nu}n^\mu n^\nu = \sigma$ (where $\sigma = \pm 1$ depending on whether n^μ is spacelike or timelike), we can construct the tensor

$$P_{\mu\nu} = g_{\mu\nu} - \sigma n_\mu n_\nu \ .$$

Of course we can raise and lower indices as usual.

(a) Show that $P^\mu{}_\nu$ projects a vector A^μ into one orthogonal to n^μ; that is, show that $P^\mu{}_\nu A^\nu$ is both orthogonal to n^μ and also unaffected by P:

$$P^\mu{}_\nu P^\nu{}_\rho A^\rho = P^\mu{}_\sigma A^\sigma \ .$$

(b) Show that $P_{\mu\nu}$ acts like the metric on vectors orthogonal to n; that is, for A^μ and B^ν orthogonal to n,

$$P_{\mu\nu}A^\mu B^\nu = g_{\mu\nu}A^\mu B^\nu \ .$$

2. Consider a two-dimensional space with coordinates $x^\mu = (x, y)$ and metric

$$ds^2 = \frac{a^2}{y^2}(\mathrm{d}x^2 + \mathrm{d}y^2) \ ,$$

where a is a fixed constant. This is one way of writing the hyperbolic plane, a maximally symmetric space with constant negative curvature. Calculate the connection coefficients, Riemann tensor, Ricci tensor, and curvature scalar, using the definitions:

$$\begin{aligned}
\Gamma^\rho_{\mu\nu} &= \frac{1}{2}g^{\rho\lambda}(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}), \\
R^\rho{}_{\sigma\mu\nu} &= \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}, \\
R_{\mu\nu} &= R^\lambda{}_{\mu\lambda\nu}, \\
R &= g^{\mu\nu}R_{\mu\nu}. \qquad (0.1)
\end{aligned}$$

Save yourself work by noting that, because of the symmetries of the Riemann tensor and the diagonal metric, the only nonvanishing components of $R^\rho{}_{\sigma\mu\nu}$ are those with $\rho \neq \sigma$ and $\mu \neq \nu$.

3. Consider a particle (not necessarily on a geodesic) which has fallen inside the event horizon, $r < 2GM$, of a Schwarzschild black hole. Use the ordinary Schwarzschild coordinates (t, r, θ, ϕ), in which the metric takes the form

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) \mathrm{d}t^2 + \left(1 - \frac{2GM}{r}\right)^{-1} \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2) \ . \tag{0.2}$$

Show that the radial coordinate must decrease at a minimum rate given by

$$\left|\frac{dr}{d\tau}\right| \geq \sqrt{\frac{2GM}{r} - 1} \ .$$

Calculate the maximum lifetime for a particle along a trajectory from $r = 2GM$ to $r = 0$. (There's an integral, which you can look up or do by clever substitution.) Express this in seconds for a black hole with mass measured in solar masses.

4. A very enthusiastic physics professor tends to wave her hands up and down rapidly when she is emphasizing a point in lecture. Let's say the frequency of handwaving is about $5\,\mathrm{s}^{-1}$. The quadrupole formula for the power radiated is

$$L = \frac{1}{5}\frac{G}{c^5}\left\langle \frac{d^3 J_{ij}}{dt^3}\frac{d^3 J^{ij}}{dt^3} \right\rangle, \tag{0.3}$$

where $\langle \cdots \rangle$ indicates an average over several oscillations. Here, J_{ij} is the reduced quadrupole tensor, $J_{ij} = I_{ij} - \frac{1}{3} I \delta_{ij}$, and the quadrupole tensor itself is

$$I_{ij} = \int \rho(x) x_i x_j \, d^3x, \tag{0.4}$$

with $\rho(x)$ being the energy density. (We are assuming a flat background spacetime, so spatial indices are raised and lowered with δ^{ij} and δ_{ij}.) How much power does she radiate in the form of gravitational waves? How does this compare to the total energy she is expending? (Use reasonable estimates for the size of the professor.)