CGWAS continuous wave exercises

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1 Sensitivity of a known pulsar $\mathcal{F}$-statistic search

Let's see where that factor of ~ 11 in sensitivity,

$$h_0 > 11\sqrt{\frac{T}{S_h}}, \tag{1}$$

comes from. Sort of. This is the story told in the first LIGO continuous waves paper: Abbott *et al.*, Phys. Rev. D **69**, 082004 (2004); although if you read carefully you'll see that it's answering a slightly different question.

Computational note: I usually use MATHEMATICA, which can be duplicated (for one-liners) at `wolframalpha.com`. In Python, you'll probably want something like:

```
from scipy.stats import chi2
chi2.cdf(x,4)
```

Also `ncx2` for noncentral χ^2 and so on. I don't use Python much, so if you google those you'll know as much as I do.

(a) First the noise. In gaussian noise with no signal, $2\mathcal{F}$ is drawn from a χ^2 distribution with 4 degrees of freedom. What value of $2\mathcal{F}$ corresponds to a 1% false alarm rate? (For reference, the mean of the distribution is 4.)

(b) Now the signal. Find the noncentrality parameter that gives you a 10% false dismissal rate at the above value of $2\mathcal{F}$.

(c) That noncentrality parameter is $h_0^2 S_h/T$ times a mess of trig functions, let's call it f. You saw earlier that the sky average of the interferometer beam patterns $F_+^2 + F_\times^2$ is 2/5. Here the average of f is 4/25, not 2/5, because we are averaging over all possible directions of the neutron star's rotation axis as well as sky positions. Why? Don't calculate—instead imagine yourself on the neutron star and think what it looks like from there. Then multiply your answer to (b) by 25/4 and take the square root to find a few significant figures beyond "11 point something" in the sensitivity factor.

Going further: If you really want to see what a mess those trig functions are, and see various averages calculated with no shortcuts, see the the paper that introduced the $\mathcal{F}$-statistic: Jaranowski, Królak, and Schutz, Phys. Rev. D **58**, 063001 (1998).

2 Elastic deformations of neutron stars

Let's consider a mass quadrupole "mountain" (more like a continent in horizontal extent!) on a neutron star, and why the structure factor doesn't matter so much. Neglecting the (Newtonian) self-gravity of the perturbation, we have

$$\left|I^{2,2}\right| = \frac{8\pi}{15}\sqrt{\frac{32\pi}{5}}\int_0^R dr\,\frac{\mu r^3}{g}\left(48 - 14U + U^2 - r\frac{dU}{dr}\right)\sigma, \tag{2}$$

where μ is the shear modulus of the matter, g is the local gravitational acceleration, and

$$U \equiv 2 + \frac{r}{g}\frac{dg}{dr}. \tag{3}$$

These are all functions of r; the stellar radius R is not. In general the dimensionless strain σ could be a function of position, but it is usually treated as constant. Let's consider the two extremes of incompressible and infinitely compressible matter.

(a) The latter is a decent approximation to a normal neutron star: The solid part ($\mu > 0$) is all in the crust, which is the outermost layer of thickness ΔR. In the infinitely compressible limit, all the mass of the star is interior to the crust. In this limit, for constant μ, find $\left|I^{2,2}\right|$ as a function of M, R, and σ. You can assume that $\Delta R/R$ is small.

(b) Now take the incompressible limit: The density is constant. In some quark star models, the solid part is basically the whole star. Under these assumptions, find $\left|I^{2,2}\right|$ as a function of M, R, and σ. It should be almost identical to your previous answer for $\Delta R/R \approx 1/10$ as in a neutron star.

(c) Plug in some numbers for part (a). Nowadays people think the maximum σ is about 10^{-1} before the material breaks. A big neutron star is about $M = 2\ M_\odot$ and $R = 12$ km, with a 1 km thick crust. An appropriately averaged shear modulus μ is about $4 \times 10^{29}\,\mathrm{erg/cm^3}$. Your number should be much less than the fiducial moment of inertia $10^{45}\,\mathrm{g\,cm^2}$ people often use for comparison.

Going further: The above simplified approach is described in Owen, Phys. Rev. Lett. **95**, 211101 (2005) and Owen, Phys. Rev. D **82**, 104002 (2010).

3 R-modes

For a slowly rotating star in Newtonian gravity, a small amplitude r-mode is a velocity perturbation that looks like

$$\delta\mathbf{v} = \alpha\Omega R(r/R)^\ell \mathbf{Y}^B_{\ell\ell} e^{i\omega t}, \tag{4}$$

where

$$\mathbf{Y}^B_{\ell m} = \frac{\mathbf{r}\times\nabla Y_{\ell m}}{\sqrt{\ell(\ell+1)}} \tag{5}$$

is a magnetic type vector spherical harmonic (so called because it is divergence free) and $Y_{\ell m}$ is a scalar spherical harmonic. Also α is an amplitude parameter,

Ω is the angular spin frequency of the star, R is the star's radius, and ω is the angular frequency of the mode as seen in an inertial frame of reference. Both types of spherical harmonic are, by construction, orthonormal when integrated over solid angle.

(a) Find the perturbed energy of the r-mode by integrating over the whole star. (Hint: it's all kinetic.) For this problem you can assume the star is all fluid (it turns out the crust mostly moves along with the fluid anyway). Make no other assumption about the equation of state, so your answer is some constants times an integral involving $\rho(r)$. (For slow rotation, you can also neglect the angular dependence of the density of the background rotating star.) One subtlety: Since we are using a complex perturbation, as is common in the literature, $\delta\mathbf{v} \cdot \delta\mathbf{v}$ is replaced by $\delta\mathbf{v} \cdot \delta\mathbf{v}^*$.

(b) Gravitational waves come from current multipoles

$$S^{\ell m} = \frac{-32\pi}{(2\ell+1)!!}\sqrt{\frac{\ell+2}{2(\ell-1)}} \int d^3r \rho r^\ell \mathbf{v}/c \cdot \left(\mathbf{Y}^B_{\ell m}\right)^* \tag{6}$$

as well as the usual mass multipoles

$$I^{\ell m} = \frac{16\pi}{(2\ell+1)!!}\sqrt{\frac{(\ell+1)(\ell+2)}{2\ell(\ell-1)}} \int d^3r \rho r^\ell Y^*_{\ell m}. \tag{7}$$

In both of these expressions 5!! means $5 \cdot 3 \cdot 1$ and so on. Find the $\ell = m = 2$ multipoles produced by the r-mode above.

(c) Plug in numbers for the current quadrupole. Use the same stellar parameters as in the previous problem. Also note that

$$\tilde{J} \equiv \frac{1}{MR^4} \int_0^R dr\, \rho r^6 \tag{8}$$

depends only weakly on the equation of state; here you can use 0.01635. For the spin frequency use 30 Hz, which is about right for the Crab pulsar.

Going further: More detail on r-mode driving, damping, and evolution timescales can be found in: Lindblom, Owen, and Morsink, Phys. Rev. Lett. **80**, 4843 (1998) and Owen *et al.*, Phys. Rev. D **58**, 084020 (1998).