CGWAS 2015: SNR from a compact binary inspiral
Alan Weinstein, July 2015

Consider a binary neutron star (BNS) system, composed of two neutron stars, each of mass 1.4 $M_\odot$, in a tight circular orbit around their common center of mass. Quaestions 1-9 below require only a knowledge of freshman Newtonian physics.

1. When their orbital period is 0.1 s, what is the frequency of GWs that they emit in Hz?
2. What is their radial separation a in km? What is their velocity in km/s, and what is (v/c)?
3. What is their kinetic energy and gravitational potential energy in ergs? What is their orbital angular momentum?
4. What is the Innermost Stable Circular Orbit (ISCO)? What is its value in terms of fundamental constants? (Answer: $a_{ISCO} = 6GM/c^2$). In km?
5. What is the orbital period and frequency at ISCO? What is the frequency of GWs that they emit in Hz?
6. What is their radial separation in km at ISCO? What is their velocity in km/s, and what is (v/c)?
7. What is their kinetic energy and gravitational potential energy in ergs at ISCO? What is their orbital angular momentum?
8. During the inspiral between an orbital period of 0.1 s and ISCO, how much energy has been lost, and in what form?
9. What is the gravitational wave energy flux at 1 Mpc due to this inspiral? (Hint: all you need is conservation of energy).
10. From GR, we can compute the strain in the quadrupole approximation, and in the frequency domain with the stationary phase approximation (extra credit; or better, wait for Duncan's talk):
$$\tilde{h}(f) = \left(\frac{5\pi}{24}\right)^{1/2} \frac{G^2\mathcal{M}^2}{c^5 D} \left(\pi G\mathcal{M}f/c^3\right)^{-7/6} e^{-i\Psi(f)};$$
$$\Psi(f) = 2\pi f t_c - 2\phi_c - \frac{\pi}{4} + \frac{3}{128}\left(\pi G\mathcal{M}f/c^3\right)^{-5/3},$$
where $\mathcal{M} = \mu^{3/5}M^{2/5} = \eta^{3/5}M$ is the *chirp mass*, D is the *luminosity distance*, t_c is the time at coalescence, and ϕ_c is the phase at coalescence.
11. Note that, at the level of this approximation, both the amplitude and phase of the waveform depend on M_1 and M_2 only in the combination $\mathcal{M}$. So if the phase evolution is measured by the detector, the chirp mass and thus the amplitude is known; the source becomes a *standard siren* (*not* a *standard candle*; why?). One can then determine the luminosity distance D to the source, even if many Mpc away. (This is very hard to do in traditional optical astronomy).

12. If the source is face-on and directly overhead a GW detector with strain-equivalent noise power $S(f)$, the ideal signal-to-noise ratio (SNR, ρ) is given by:

$$\rho^2 = 4 \int_{f_{min}}^{f_{max}} \frac{\tilde{h}^*(f)\tilde{h}(f)}{S(f)} df.$$

For an initial-LIGO-like detector, the noise power (in strain$^2/Hz$) is the incoherent sum of three main contributions: seismic (S_{sei}), thermal (S_{th}), and shot-noise (S_{sh}). A crude model of these noise power spectra is:

$$S_{sei} = \left(1.0 \times 10^{-36}\right) \left(\frac{f}{20\,Hz}\right)^{-28};$$
$$S_{th} = \left(1.0 \times 10^{-44}\right) \left(\frac{f}{62\,Hz}\right)^{-4};$$
$$S_{sh} = \left(1.0 \times 10^{-46}\right) + \left(1.0 \times 10^{-42}\right) \left(\frac{f}{7900 Hz}\right)^{2}.$$

Plot the amplitude spectral density (ASD $= \sqrt{S(f)}$) on a log-log scale.

13. Plot the $h(f)$ spectrum on top of that. But be careful! Do they have the same units? How can you correct for the unit mismatch? (Answer: plot $h(f)\sqrt{f}$).

14. Compute the SNR of our inspiral signal at a distance of $D = 1$ Mpc.

15. At what distance (in Mpc) can we see such a signal with SNR of 8 (the *BNS horizon distance*)? (Do you know why we set the SNR detection threshold to 8?).

16. If the source is not face-on and directly overhead, a GW detector will respond with reduced sensitivity (the *antenna pattern*). We can take a volume average over sky location and orientation of the source, expressed as the ratio of *BNS horizon distance* to *BNS average range*; it is is 2.26 (ultra extra credit if you can compute that!).

17. It is estimated (http://arxiv.org/abs/1003.2480, Table IV) that there is roughly one binary neutron star merger per Mpc^3 per Myr (averaged over sky location and orientation). How many do we expect to see in initial LIGO per year?

18. OK, now repeat the whole exercise for BBH ($M_1 = M_2 = 10\,M_\odot$), and NSBH ($M_1 = 1.4\,M_\odot, M_2 = 10\,M_\odot$).

19. Plot the horizon distance versus total mass (up to, say, 500 $M_\odot$) for equal-mass systems.

20. Extra credit, if you know how: use waveforms that include the merger and ringdown (not really necessary for BNS; why?); and use the Advanced LIGO noise spectrum. How many mergers will Advanced LIGO see per year?

The predicted compact binary inspiral signal amplitude at the detector is:

$$\tilde{h}(f) = \mathcal{A}\left(\frac{5\pi}{24}\right)^{1/2}\frac{G^2\mathcal{M}^2}{c^5 D}\left(\pi G\mathcal{M}f/c^3\right)^{-7/6} e^{-i\Psi(f)};$$

$$\Psi(f) = 2\pi f t_c - 2\phi_c - \frac{\pi}{4} + \frac{3}{128}\left(\pi G\mathcal{M}f/c^3\right)^{-5/3},$$

$\mathcal{M} = \eta^{3/5}M$ is the *chirp mass*,
$M = m_1 + m_2$ is the total mass,
$\eta = m_1 m_2/M^2$ is the symmetric mass ratio $\in [0, 1/4]; \eta = 1/4$ for $m_1 = m_2$
D is the *luminosity distance* (in Mpc),
$\mathcal{A}$ is an antenna factor $\in [0, 1]$;
the volume-averaged antenna factor $\langle\mathcal{A}\rangle = 1/2.26 = 0.4425$.

The *ideal* signal-to-noise ratio (SNR, or ρ):

$$\rho = \sqrt{4\int_{f_{min}}^{f_{max}} \frac{\tilde{h}^*(f)\tilde{h}(f)}{S(f)} df}$$

where $S(f)$ is the detector noise amplitude spectral density (ASD). The SNR goes like $1/D$.

We think we can confidently detect a signal with $\rho \geq 8$.
What luminosity distance does this correspond to, in Mpc?
This is called the binary inspiral range.
The result depends only on $S(f)$ and $\mathcal{M}$.
For our canonical standard siren, Binary Neutron Star (BNS) inspiral,
$m_1 = m_2 = 1.4M_\odot$ and $\mathcal{M} = 1.22$. This gives the *BNS range.*

An optimally located and oriented source has $\mathcal{A} = 1$;
the corresponding luminosity distance is called the *inspiral horizon distance.*