Detection of Compact Binaries

In the case of CBCs, we know

$$h(t) = \frac{1}{D} A(t) \cos\left(\phi(t) - \theta\right)$$

where $A(t) \propto f^{2/3}$ and $\phi(t)$ comes from PN, EOB, etc.

$\Rightarrow$ We know the form of the signal (c.f. burst searches)

Our job is to decide if the detector output

$$s(t) = \begin{cases} n(t) & \text{noise alone} \\ n(t) + h(t) & \text{detection!} \end{cases}$$

What does the noise $n(t)$ look like? The simplest model is that the noise is stationary and Gaussian, with zero mean, so probability of obtaining $n(t)$ is

$$p(n) = K \exp\left[-\tfrac{1}{2}(n|n)\right]$$

where

$$(a|b) \equiv \int_{-\infty}^{\infty} df \, \frac{\tilde{a}^*(f)\tilde{b}(f) + \tilde{a}(f)\tilde{b}^*(f)}{S_n(f)}$$

and

$$\tilde{a}(f) = \int_{-\infty}^{\infty} a(t)\, e^{2\pi i f t}\, dt.$$

The quantity $S_n(f)$ is defined by

$$\langle \tilde{n}(f)\,\tilde{n}(f') \rangle = \tfrac{1}{2} S_n(f)\, \delta(f-f')$$

and is called the (one sided) power spectral density of the noise.

$S_n(f)$ tells us how much noise power we have in a given frequency bin - see Blandford & Thorne for more discussion.

What we want to compute is the probability that the data contains a signal h given that we observe the output data s.

Bayes' theorem says

$$p(h|s) = \frac{p(h)\,p(s|h)}{p(s)}$$

$p(s|h)$ is the probability of obtaining the detector output, given that a signal is present; it is called the likelihood

$$p(s) = \underbrace{p(h)\,p(s|h)}_{\text{signal}} + \underbrace{p(0)\,p(s|0)}_{\text{no signal}}$$

substituting this into Bayes' theorem and dividing the numerator and denominator by $p(h)p(s|0)$, we obtain

$$p(h|s) = \frac{p(s|h)/p(s|0)}{\underbrace{p(s|h)/p(s|0)}_{=\,\Lambda\text{, the likelihood ratio}} + p(0)/p(h)}$$

So
$$p(h|s) = \frac{\Lambda}{\Lambda + p(0)/p(h)}$$

Now
$$p(0|s) = 1 - p(h|s)$$

So
$$\frac{p(h|s)}{p(0|s)} = \Lambda \frac{p(h)}{p(0)}$$

depends on data

priors that are independent of the data – constant

So we want to construct Λ for a CBC signal, given the data.

$$\Lambda = \frac{p(s|h)}{p(s|0)} = \exp\left[(s|h) - \tfrac{1}{2}(h|h) \right]$$

depends on data

constant

So we can construct $(s|h)$ and threshold on this quantity.

In practice
$$\Lambda(\theta) = p(\theta) \exp\left[\frac{1}{D}\left(s \,|\, A(t)\cos(\phi(t)-\theta)\right) - \frac{1}{2}\frac{1}{D^2}(h|h) \right]$$

$1/2\pi$

Need to integrate out the phase θ

Now we write

$$\left(s \middle| A(t)\cos(\phi(t)-\theta)\right) = \cos\theta\,\underbrace{\left(s \middle| A\cos 2\phi\right)}_{|z|\cos\Phi \,=\, x} + \sin\theta\,\underbrace{\left(s \middle| A\sin 2\phi\right)}_{|z|\sin\Phi \,=\, y}$$

$$|z| = \sqrt{x^2+y^2}$$

$$\Lambda = \frac{1}{2\pi}\int_0^{2\pi} \exp\left[\frac{1}{D}|z|\cos(\Phi-\theta) - \frac{1}{2D^2}(h|h)\right] d\theta$$

$$= I_0\left(D^{-1}|z|\right) e^{-\frac{1}{2r^2}(h|h)}$$

Modified Bessel fn of first kind

Monotonic in z

We threshold on $|z| = \sqrt{(s|h_c)^2 + (s|h_c)^2}$

Define a normalised signal to noise ratio by

$$\rho^2 = \frac{|z|^2}{\sigma^2} \qquad \text{where } \sigma^2 = (h_c|h_c) = (h_s|h_s)$$

What about time of arrival?

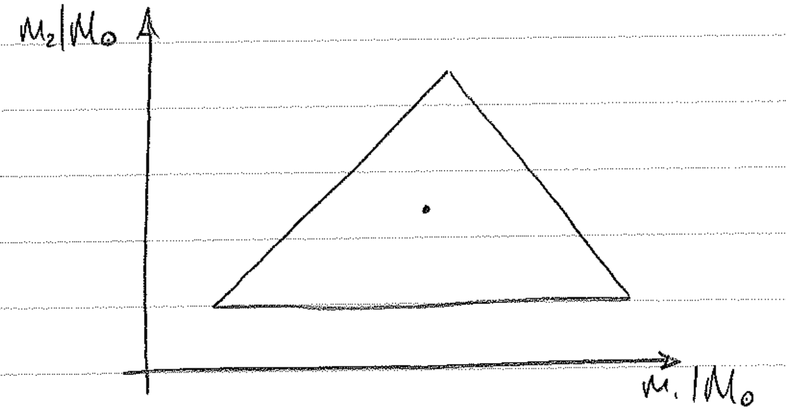
$$\rho(t) = \frac{1}{\sigma}\sqrt{(s|h_c)^2 + (s|h_s)^2} \qquad (s|h_c) = 2\int_{-\infty}^{\infty} df\, e^{2\pi i f t}\,\frac{\tilde{s}(f)\,\tilde{h}^*(f)}{S_n(f)}$$

Note that we need $\tilde{h}(f)$ the template in the frequency domain. Can either compute $h(t)$ and Fourier transform, or write $\tilde{h}(f)$ from PN directly in the frequency domain (see exercises) as

$$\tilde{h}(f) = A(f)\, e^{i\Psi(f)}$$

TaylorF2 or SPA.

Searching for signals



Define the match as

$$M = \max_{t_c, \phi_c} \frac{(h_1|h_2)}{\sqrt{(h_1|h_1)}\,\sqrt{(h_2|h_2)}}$$

Now locally $\quad 1 - \big(h(\lambda)\,|\,h(\lambda+\delta\lambda)\big) = g_{ij}\,\delta\lambda^i\,\delta\lambda^j$

$$g_{ij} = -\frac{1}{2}\,\frac{\partial^2 (h(\lambda)|h(\lambda))}{\partial\lambda^i\,\partial\lambda^j}$$

And use this metric to place a bank of templates so that

$1 - M \leq 0.03$ $\Rightarrow$ 3% loss in SNR

Note that m_1 and m_2 are not the best coordinates for the bank space. In practice we use

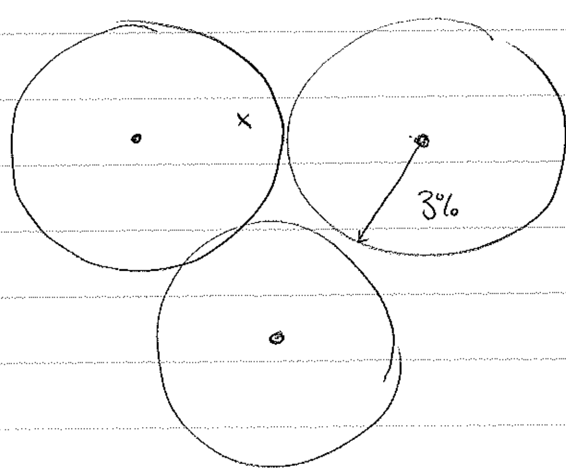
$$\tau_0 = \frac{5}{256\pi f_{low}\eta}\left(\frac{\pi G M f_{low}}{c^3}\right)^{-5/3}$$

$$\tau_3 = \frac{5}{8 f_{low}\eta}\left(\frac{\pi G M f_{low}}{c^3}\right)^{-2/3}$$

called the "chirp times"

Here f_{low} is the low frequency cutoff.

Notice that g_{ij} depends on $S_n(f)$. The better the low-frequency sensitivity, the more templates.



use hexagonal placement to grid up space

For stellar mass BH and NS $\sim 10^5$ templates (non-spinning!)

The detection problem becomes

$$\text{Compute} \quad \rho^2(t) = \frac{1}{\sigma^2}(\rho_c^2 + \rho_s^2)$$

and threshold. To measure the distance to the signal set $D = 1\,\mathrm{Mpc}$ in the template waveform then

$$D = \frac{\sigma}{\rho}$$

is the effective distance to the source.

The distribution of the SNR in Gaussian noise is

$$p(|\rho|) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\rho^2}{2}\right)$$

We are interested in the probability of getting $|\rho| > \rho^*$ in a single trial, which is obtained by integrating $p(|\rho|)$ from ρ^* to infinity, i.e.

$$p(|\rho| > \rho^*) = \mathrm{erfc}\left[\rho^*/\sqrt{2}\right]$$

$$\rho^* = \begin{cases} 5 & \rightarrow \quad 10^{-7} \\ 6 & \quad 10^{-9} \\ 7 & \quad 10^{-12} \end{cases}$$

How many trials in a year? For an inspiral signal, the autocorrelation time is $\sim 3\,\mathrm{ms}$, so 10^{10} independent trials in 1 year of data. Need $\sim 10^4$ independent templates to cover mass space

So for a false alarm rate of 1/100 years, we need

$$p(|\rho| > \rho^*) = 10^{-17}$$

or $\rho^* = 8.6$.

But we have multiple detectors! Construct a network SNR

$$\rho = \sqrt{\rho_H^2 + \rho_L^2}$$ and demand same signal

If we have 10^{10} time samples, 10^4 templates, 10 different sky locations, need $\rho > 9.5$ for 1/100 years.

Worse? No corresponds to $\rho \sim 6.7$ in each detector.

Problem!

$n(t)$ is <u>not</u> stationary and Gaussian.

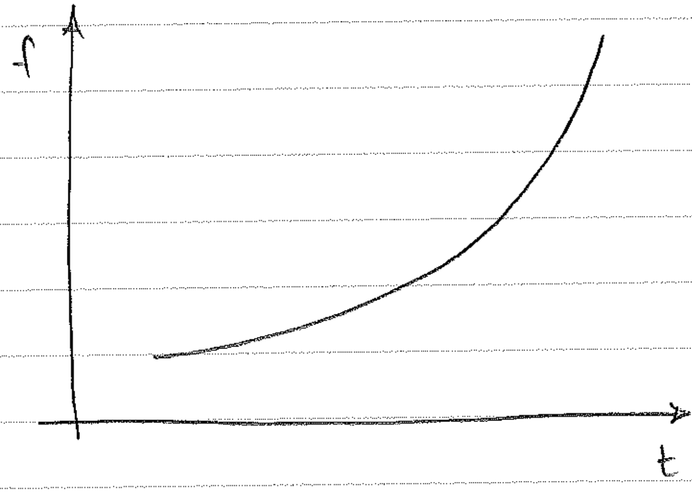
Need additional statistics. Why? What is $\rho^2(t)$ if

$$s(t) = n(t) + \delta(t - t_d)$$

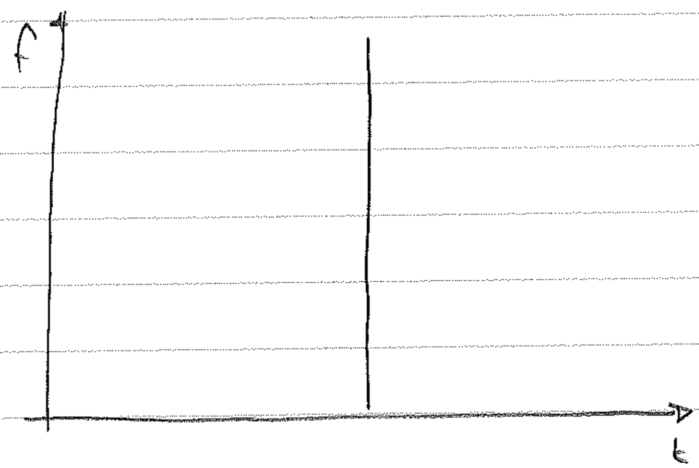
This is the impulse response of the matched filter.

$\Rightarrow$ causes peaks of high snr that are not due to signal!

We know that for a signal



but a glitch looks like this



How can we discriminate? Divide $h(t)$ into p bins of equal power

Construct

$$\chi^2 = \frac{1}{p} \sum_{l=1}^{p} \left[\frac{\rho}{p} - \rho_l \right]^2$$

This is asking " do we get $\frac{1}{p}$th the power from the p^{th} bin? "

This statistic is χ^2 distributed with $2p-2$ degrees of freedom

signal $\Rightarrow$ χ^2 small , glitch $\Rightarrow$ χ^2 big.

So weight the SNR by the χ^2

$$\rho_{new} = \begin{cases} \rho & \text{if } \chi^2 \leq n_{dof} \\ \rho\left[\frac{1}{2}\left(1+\left(\frac{\chi^2}{n_{dof}}\right)^3\right)\right]^{-1/6} & \chi^2 > n_{dof} \end{cases}$$

and use this as the detection statistic.

Now the map between ρ_{new} and false-alarm rate is complicated!

- Don't have a good model for the non-Gaussian noise
- Complicated detection statistic
- Correllations between templates

So we need to measure it.

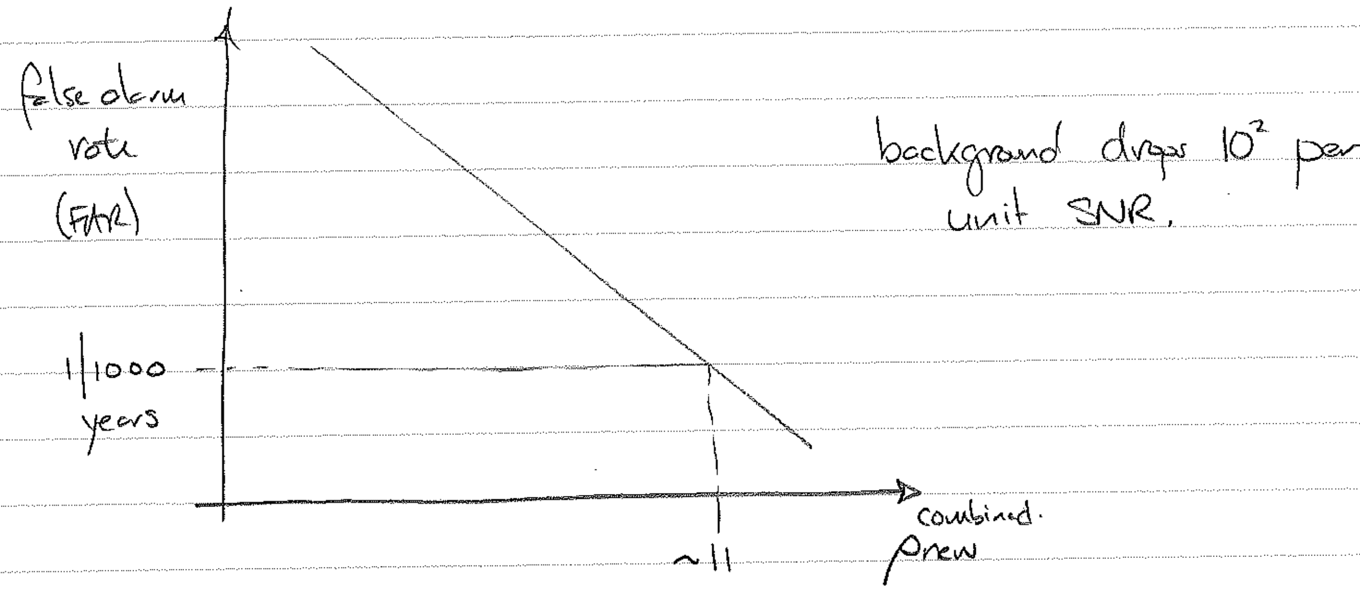
A signal is coincident if it has the "same" mass parameters and arrives within light travel time + measurement error.

Shift the data by more than the light travel time

$\Rightarrow$ any coincidences are due to noise alone

This gives a measure of the rate of noise coincidences; our background rate. Repeat this many times with many different shifts to build up statistics.

Obtain



So combined new SNR of 11 would have a FAR of 1/1000 years

$\Rightarrow$ detection!